Discrete Mathematics 311 (2011) 2417–2422

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Cyclic orthogonal double covers of 4-regular circulant graphs

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ARTICLE INFO

Article history:
Received 31 May 2010
Received in revised form 14 June 2011
Accepted 20 June 2011
Available online 30 July 2011

Keywords:
Orthogonal double covers of graphs
Labellings of graphs
Circulant graphs

ABSTRACT

An *orthogonal double cover* (ODC) of a graph H is a collection $\mathcal{G} = \{G_v : v \in V(H)\}$ of $|V(H)|$ subgraphs of H such that every edge of H is contained in exactly two members of $\mathcal{G}$ and for any two members G_u and G_v in $\mathcal{G}$, $|E(G_u) \cap E(G_v)|$ is 1 if u and v are adjacent in H and it is 0 if u and v are nonadjacent in H. An ODC $\mathcal{G}$ of H is *cyclic* (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$. In this paper, we are concerned with CODCs of 4-regular circulant graphs.

1. Introduction

Let H be any graph and let $\mathcal{G} = \{G_1, G_2, \ldots, G_{|V(H)|}\}$ be a collection of $|V(H)|$ subgraphs of H. $\mathcal{G}$ is a *double cover* (DC) of H if every edge of H is contained in exactly two members in $\mathcal{G}$. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{G}$ is a *DC* of H by G. If $\mathcal{G}$ is a DC of H by G, then $|V(H)|\,|E(G)| = 2|E(H)|$.

A DC $\mathcal{G}$ of H is an *orthogonal double cover* (ODC) of H if there exists a bijective mapping $\phi : V(H) \to \mathcal{G}$ such that for every choice of distinct vertices u and v in $V(H)$, $|E(\phi(u)) \cap E(\phi(v))|$ is 1 if $uv \in E(H)$ and is 0 otherwise. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{G}$ is an *ODC* of H by G.

An *automorphism* of an ODC $\mathcal{G} = \{G_1, G_2, \ldots, G_{|V(H)|}\}$ of H is a permutation $\pi : V(H) \to V(H)$ such that $\{\pi(G_1), \pi(G_2), \ldots, \pi(G_{|V(H)|})\} = \mathcal{G}$, where for $i \in \{1, 2, \ldots, |V(H)|\}$, $\pi(G_i)$ is a subgraph of H with $V(\pi(G_i)) = \{\pi(v) : v \in V(G_i)\}$ and $E(\pi(G_i)) = \{\pi(u)\pi(v) : uv \in E(G_i)\}$. An ODC $\mathcal{G}$ of H is *cyclic* (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$, the set of all automorphisms of $\mathcal{G}$.

Throughout the article we make use of the usual notation: K_n for the complete graph on n vertices, $K_{m,n}$ for the complete bipartite graph with independent sets of sizes m and n, P_n for the path on n vertices, C_n for the cycle on n vertices, $G + H$ for the disjoint union $G \cup H$ of G and H, and ℓG for ℓ disjoint copies of G.

Let $n_1, n_2, \ldots, n_r$, $r \geq 1$, be integers, n_1, $n_r \geq 1$ and $n_i \geq 0$ for $i \in \{2, 3, \ldots, r-1\}$. The *caterpillar* $C_r(n_1, n_2, \ldots, n_r)$ is the tree obtained from the path $P_r := x_1 x_2 \ldots x_r$ by joining vertex x_i to n_i new vertices, $i \in \{1, 2, \ldots, r\}$.

For a sequence $\{d_1, d_2, \ldots, d_k\}$ of positive integers with $1 \leq d_1 < d_2 < \cdots < d_k \leq \lfloor \frac{n}{2} \rfloor$, the *circulant graph* $Circ(n; \{d_1, d_2, \ldots, d_k\})$ has vertex set $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$; two vertices x and y are adjacent if and only if $x - y \equiv \pm d_i \pmod n$ for some i, $i \in \{1, 2, \ldots, k\}$. For an edge xy in $Circ(n; \{d_1, d_2, \ldots, d_k\})$, the *length* of xy is $\min\{|x-y|, n-|x-y|\}$.

Given two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of the same length ℓ in $Circ(n; \{d_1, d_2, \ldots, d_k\})$, the *rotation-distance* $r(\ell)$ between e_1 and e_2 is $r(\ell) = \min\{r_1, r_2 : (u_1+r_1)(v_1+r_1) = e_2, (u_2+r_2)(v_2+r_2) = e_1\}$, where addition is reduced modulo n. Observe that if $r(\ell) = \ell$, then the edges e_1 and e_2 are adjacent; if $r(\ell) \neq \ell$, then e_1 and e_2 are nonadjacent.

Consider the complete graph $K_n = Circ(n; \{1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor\})$. The authors of [3] introduced the notion of an orthogonal labelling. Given a graph $G = (V, E)$ with $n-1$ edges, a 1–1 mapping $\psi : V \to \mathbb{Z}_n$ is an *orthogonal labelling* of G if: (i) for

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doi:10.1016/j.disc.2011.06.021

every $\ell \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, G contains exactly *two* edges of length ℓ, and exactly *one* edge of length $\frac{n}{2}$ if n is even, and (ii)

$$\left\{r(\ell) : \ell \in \left\{1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor\right\}\right\} = \left\{1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor\right\}.$$

The following theorem of Gronau et al. [3] relates CODCs of K_n and orthogonal labellings.

Theorem 1.1 (*[3]*). *A CODC of K_n by a graph G exists if and only if there exists an orthogonal labelling of G.*

Sampathkumar and Simaringa called an orthogonal labelling an orthogonal $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$-labelling and generalized it to an orthogonal $\{d_1, d_2, \ldots, d_k\}$-labelling, where $\{d_1, d_2, \ldots, d_k\}$ is a sequence of positive integers with $1 \le d_1 < d_2 < \cdots < d_k \le \lfloor \frac{n}{2} \rfloor$.

I. *Either* n is odd *or* n is even and $d_k \ne \frac{n}{2}$:

Given a subgraph G of $Circ(n; \{d_1, d_2, \ldots, d_k\})$ with $2k$ edges, a labelling of G, in $\mathbb{Z}_n$, is an *orthogonal* $\{d_1, d_2, \ldots, d_k\}$-*labelling* of G if: (i) for every $\ell \in \{d_1, d_2, \ldots, d_k\}$, G contains exactly *two* edges of length ℓ, and (ii) $\{r(\ell) : \ell \in \{d_1, d_2, \ldots, d_k\}\} = \{d_1, d_2, \ldots, d_k\}$.

II. n is even and $d_k = \frac{n}{2}$:

Given a subgraph G of $Circ(n; \{d_1, d_2, \ldots, d_{k-1}, \frac{n}{2}\})$ with $2k - 1$ edges, a labelling of G, in $\mathbb{Z}_n$, is an *orthogonal* $\{d_1, d_2, \ldots, d_{k-1}, \frac{n}{2}\}$-*labelling* of G if: (i) for every $\ell \in \{d_1, d_2, \ldots, d_{k-1}\}$, G contains exactly *two* edges of length ℓ, and G contains exactly *one* edge of length $\frac{n}{2}$, and (ii) $\{r(\ell) : \ell \in \{d_1, d_2, \ldots, d_{k-1}\}\} = \{d_1, d_2, \ldots, d_{k-1}\}$.

The following theorem, of Sampathkumar and Simaringa, is a generalization of Theorem 1.1. The proof of Theorem 1.2 is similar to that of Theorem 1.1.

Theorem 1.2. *A CODC of $Circ(n; \{d_1, d_2, \ldots, d_k\})$ by a graph G exists if and only if there exists an orthogonal $\{d_1, d_2, \ldots, d_k\}$-labelling of G.*

For results on ODCs of graphs, see [2], a survey by Gronau et al.

In [4], Hartmann and Schumacher proved the following. (i) Let H be a 2-regular graph. There exists an ODC of H by $2K_2$ with three exceptions for H: C_3, C_4 and $2C_3$. (ii) Let H be a 3-regular graph containing a 1-factor and without a component isomorphic to K_4. There exists an ODC of H by P_4. (iii) Let H be a 3-regular graph containing a 1-factor and $|V(H)| \ge 24$. There exists an ODC of H by $P_3 + K_2$.

Let Γ be a finite group and $A \subseteq \Gamma$ be a subset of Γ such that $A^{-1} = A$ and $1 \notin A$. The Cayley graph $Cay(\Gamma, A)$ has vertex set Γ and edge set $\{\{x, ax\} : x \in \Gamma, a \in A\}$.

In [5], Scapellato et al. proved the following. (i) All 3-regular Cayley graphs, except K_4, have ODCs by P_4. (ii) All 3-regular Cayley graphs on Abelian groups, except K_4, have ODCs by $P_3 + K_2$. (iii) All 3-regular Cayley graphs on Abelian groups, except K_4 and the 3-prism (Cartesian product of C_3 and K_2), have ODCs by $3K_2$.

The above results on ODCs of graphs with lower degrees motivates us to consider ODCs of 4-regular circulant graphs. In this paper, we have completely settled the existence problem of CODCs of 4-regular circulant graphs. Note that, for ODCs of such graphs by a graph G, G has to have four edges.

Denote by H the graph with $V(H) = \{v_1, v_2, v_3, v_4\}$ and $E(H) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3\}$. The other terminology not defined here can be found in [1].

2. 4-regular circulant graphs

Suppose that $1 \le d_1 < d_2 \le \lfloor \frac{n-1}{2} \rfloor$ and let G be any simple graph with four edges. Our aim is to find a CODC of the circulant graph $Circ(n; \{d_1, d_2\})$ by G. As $|E(G)| = 4$, $G \in \{4K_2, P_3 + 2K_2, 2P_3, P_4 + K_2, K_{1,3} + K_2, C_3 + K_2, P_5, C_2(1, 2), K_{1,4}, C_4, H\}$. By Theorem 1.2, we have to find an orthogonal $\{d_1, d_2\}$-labelling of G; i.e., we have to find a labelling of the vertices of G in $\mathbb{Z}_n$ such that G contains two edges of length d_1 and two edges of length d_2, and $\{r(d_1), r(d_2)\} = \{d_1, d_2\}$. Hence, either $r(d_1) = d_1$ and $r(d_2) = d_2$ or $r(d_1) = d_2$ and $r(d_2) = d_1$.

Trivially, there exists a CODC of $Circ(n; \{d_1, d_2\})$ by $K_{1,4}$.

Theorem 2.1. *There is no CODC of $Circ(n; \{d_1, d_2\})$ by $K_{1,3} + K_2$.*

Proof. Suppose that there exists a CODC of $Circ(n; \{d_1, d_2\})$ by $K_{1,3} + K_2$. If $r(d_1) = d_1$ and $r(d_2) = d_2$, then there is a decomposition (a partition of the edge set) of $K_{1,3} + K_2$ by P_3; if $r(d_1) = d_2$ and $r(d_2) = d_1$, then there is a decomposition of $K_{1,3} + K_2$ by $2K_2$. In both cases, we have obtained a contradiction. □

Theorem 2.2. *There is no CODC of $Circ(n; \{d_1, d_2\})$ by $C_3 + K_2$.*

Proof. Similar to that for Theorem 2.1. □

Theorem 2.3. *Suppose that $n \ge 8$. A CODC of $Circ(n; \{d_1, d_2\})$ by $4K_2$ exists if and only if $(n, d_1, d_2) \notin \{(8, 1, 3), (9, 1, 3), (9, 2, 3), (9, 3, 4)\}$.*

Proof. If there exists a CODC of $Circ(n; \{d_1, d_2\})$ by $4K_2$, then $r(d_1) = d_2$ and $r(d_2) = d_1$.

First, assume that $(n, d_1, d_2) \notin \{(8, 1, 3), (9, 1, 3), (9, 2, 3), (9, 3, 4)\}$.

Case 1. $d_1 + d_2 \neq \lfloor \frac{n}{2} \rfloor$, $d_1 + d_2 \neq \lceil \frac{n}{2} \rceil$ and $d_2 \neq \lfloor \frac{n}{2} \rfloor$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_2, d_1 + d_2\}$; ones of length d_2 are $\{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + d_2\}$ and $\{\lceil \frac{n}{2} \rceil + d_1, \lceil \frac{n}{2} \rceil + d_1 + d_2\}$.

Case 2. n is even and $d_1 + d_2 = \frac{n}{2}$.

Subcase 2.1. $(d_1, d_2) \neq (1, \frac{n-2}{2})$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_2, \frac{n}{2}\}$; ones of length d_2 are $\{\frac{n+2}{2}, \frac{n+2}{2} + d_2\}$ and $\{\frac{n+2}{2} + d_1, 1\}$.

Subcase 2.2. $(d_1, d_2) = (1, \frac{n-2}{2})$.

By hypothesis, $n \geq 10$. Edges of length 1 are $\{0, 1\}$ and $\{\frac{n-2}{2}, \frac{n}{2}\}$; ones of length $\frac{n-2}{2}$ are $\{2, \frac{n+2}{2}\}$ and $\{3, \frac{n+4}{2}\}$.

Case 3. n is odd and $d_2 = \frac{n-1}{2}$.

Subcase 3.1. $d_1 \notin \{\frac{n-3}{2}, \frac{n-1}{4}, \frac{n-3}{4}\}$.

Edges of length d_1 are $\{0, d_1\}$ and $\{\frac{n-1}{2}, \frac{n-1}{2} + d_1\}$; ones of length $\frac{n-1}{2}$ are $\{1+d_1, 1+d_1+\frac{n-1}{2}\}$ and $\{1+2d_1, 1+2d_1+\frac{n-1}{2}\}$.

Subcase 3.2. $d_1 = \frac{n-3}{2}$.

By hypothesis, $n \neq 9$. Edges of length $\frac{n-3}{2}$ are $\{\frac{n+5}{2}, 1\}$ and $\{2, \frac{n+1}{2}\}$; ones of length $\frac{n-1}{2}$ are $\{0, \frac{n-1}{2}\}$ and $\{\frac{n-3}{2}, n-2\}$.

Subcase 3.3. $d_1 = \frac{n-1}{4}$.

Edges of length $\frac{n-1}{4}$ are $\{1, \frac{n+3}{4}\}$ and $\{\frac{n+1}{2}, \frac{3n+1}{4}\}$; ones of length $\frac{n-1}{2}$ are $\{0, \frac{n-1}{2}\}$ and $\{\frac{n-1}{4}, \frac{3n-3}{4}\}$.

Subcase 3.4. $d_1 = \frac{n-3}{4}$.

Edges of length $\frac{n-3}{4}$ are $\{1, \frac{n+1}{4}\}$ and $\{\frac{n+1}{2}, \frac{3n-1}{4}\}$; ones of length $\frac{n-1}{2}$ are $\{0, \frac{n-1}{2}\}$ and $\{\frac{n-3}{4}, \frac{3n-5}{4}\}$.

Case 4. n is odd and $d_1 + d_2 = \frac{n-1}{2}$.

Subcase 4.1. $(d_1, d_2) \neq (1, \frac{n-3}{2})$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_2, \frac{n-1}{2}\}$; ones of length d_2 are $\{\frac{n+3}{2}, \frac{n+3}{2} + d_2\}$ and $\{\frac{n+3}{2} + d_1, 1\}$.

Subcase 4.2. $(d_1, d_2) = (1, \frac{n-3}{2})$.

By hypothesis, $n \neq 9$. Edges of length 1 are $\{0, 1\}$ and $\{\frac{n-3}{2}, \frac{n-1}{2}\}$; ones of length $\frac{n-3}{2}$ are $\{2, \frac{n+1}{2}\}$ and $\{3, \frac{n+3}{2}\}$.

Case 5. n is odd, $d_1 + d_2 = \frac{n+1}{2}$ and $d_2 \neq \frac{n-1}{2}$.

Subcase 5.1. $(d_1, d_2) \neq (2, \frac{n-3}{2})$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_2, \frac{n+1}{2}\}$; ones of length d_2 are $\{\frac{n+3}{2}, \frac{n+3}{2} + d_2\}$ and $\{\frac{n+3}{2} + d_1, 2\}$.

Subcase 5.2. $(d_1, d_2) = (2, \frac{n-3}{2})$.

By hypothesis, $n \neq 9$. Edges of length 2 are $\{0, 2\}$ and $\{\frac{n-3}{2}, \frac{n+1}{2}\}$; ones of length $\frac{n-3}{2}$ are $\{1, \frac{n-1}{2}\}$ and $\{3, \frac{n+3}{2}\}$.

Conversely, assume that $(n, d_1, d_2) \in \{(8, 1, 3), (9, 1, 3), (9, 2, 3), (9, 3, 4)\}$ and suppose that there exists a CODC of $Circ(n; \{d_1, d_2\})$ by $4K_2$.

Case 1. $(n, d_1, d_2) = (8, 1, 3)$.

Assume that the edges of length 1 are $\{0, 1\}$ and $\{3, 4\}$. Edges of length 3 are $\{i, i + 3\}$ and $\{i + 1, i + 4\}$ for some $i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$. For any i, the edge-induced subgraph induced by these four edges is not isomorphic to $4K_2$. Hence, there is no CODC of $Circ(8; \{1, 3\})$ by $4K_2$.

Case 2. $(n, d_1, d_2) = (9, 3, 4)$.

Assume that the edges of length 3 are $\{0, 3\}$ and $\{4, 7\}$. Edges of length 4 are $\{i, i + 4\}$ and $\{i + 3, i + 7\}$ for some $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For any i, the edge-induced subgraph induced by these four edges is not isomorphic to $4K_2$. Thus, there is no CODC of $Circ(9; \{3, 4\})$ by $4K_2$.

This completes the proof, because $Circ(9; \{1, 3\}) \cong Circ(9; \{2, 3\}) \cong Circ(9; \{3, 4\})$. □

Theorem 2.4. *Suppose that $n \geq 6$. A CODC of $Circ(n; \{d_1, d_2\})$ by $2P_3$ exists if and only if $(n, d_1, d_2) \notin \{(7, 1, 2), (7, 1, 3), (7, 2, 3), (9, 1, 3), (9, 2, 3), (9, 3, 4)\}$.*

Proof. First, assume that $(n, d_1, d_2) \notin \{(7, 1, 2), (7, 1, 3), (7, 2, 3), (9, 1, 3), (9, 2, 3), (9, 3, 4)\}$.

Case 1. $d_1 + 1 \neq d_2$ and $2d_1 + 1 \neq d_2$.

Edges of length d_1 are $\{1, 1 + d_1\}$ and $\{1 + d_1, 1 + 2d_1\}$; ones of length d_2 are $\{0, d_2\}$ and $\{d_2, 2d_2\}$ $(r(d_i) = d_i, i \in \{1, 2\})$.

Case 2. $d_1 + 1 = d_2$.

Subcase 2.1. $(d_1, d_2) \notin \{(1, 2), (2, 3), (\frac{n-3}{2}, \frac{n-1}{2})\}$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_1, 2d_1\}$; ones of length $1 + d_1$ are $\{1, 2 + d_1\}$ and $\{2 + d_1, 3 + 2d_1\}$ $(r(d_1) = d_1$ and $r(1 + d_1) = 1 + d_1)$.

Subcase 2.2. $(d_1, d_2) = (1, 2)$.

For $n \geq 8$, edges of length 1 are $\{0, 1\}$ and $\{1, 2\}$; ones of length 2 are $\{3, 5\}$ and $\{5, 7\}$ ($r(1) = 1$ and $r(2) = 2$). By hypothesis, $n \neq 7$. For $n = 6$, edges of length 1 are $\{0, 1\}$ and $\{2, 3\}$; ones of length 2 are $\{3, 5\}$ and $\{4, 0\}$ ($r(1) = 2$ and $r(2) = 1$).

Subcase 2.3. $(d_1, d_2) = (2, 3)$.

By hypothesis, $n \notin \{7, 9\}$. For $n \geq 10$ or $n = 8$, edges of length 2 are $\{n-3, n-1\}$ and $\{n-1, 1\}$; ones of length 3 are $\{0, 3\}$ and $\{3, 6\}$ ($r(2) = 2$ and $r(3) = 3$).

Subcase 2.4. $(d_1, d_2) = (\frac{n-3}{2}, \frac{n-1}{2})$.

By hypothesis, $n \notin \{7, 9\}$. For $n \geq 11$, edges of length $\frac{n-3}{2}$ are $\{0, \frac{n-3}{2}\}$ and $\{\frac{n-3}{2}, n-3\}$; ones of length $\frac{n-1}{2}$ are $\{2, \frac{n+3}{2}\}$ and $\{\frac{n+3}{2}, 1\}$ ($r(\frac{n-3}{2}) = \frac{n-3}{2}$ and $r(\frac{n-1}{2}) = \frac{n-1}{2}$).

Case 3. $2d_1 + 1 = d_2$.

Subcase 3.1. $(d_1, d_2) \notin \{(1, 3), (\frac{n-3}{4}, \frac{n-1}{2})\}$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_1, 2d_1\}$; ones of length d_2 are $\{1, 2d_1+2\}$ and $\{2d_1+2, 4d_1+3\}$ ($r(d_i) = d_i, i \in \{1, 2\}$).

Subcase 3.2. $(d_1, d_2) = (1, 3)$.

By hypothesis, $n \notin \{7, 9\}$. For $n \geq 10$, edges of length 1 are $\{0, 1\}$ and $\{1, 2\}$; ones of length 3 are $\{3, 6\}$ and $\{6, 9\}$ ($r(1) = 1$ and $r(3) = 3$). For $n = 8$, edges of length 1 are $\{0, 1\}$ and $\{3, 4\}$; ones of length 3 are $\{4, 7\}$ and $\{5, 0\}$ ($r(1) = 3$ and $r(3) = 1$).

Subcase 3.3. $(d_1, d_2) = (\frac{n-3}{4}, \frac{n-1}{2})$.

By hypothesis, $n \neq 7$. For $n \geq 11$, edges of length $\frac{n-3}{4}$ are $\{2, \frac{n+5}{4}\}$ and $\{\frac{n+5}{4}, \frac{n+1}{2}\}$; ones of length $\frac{n-1}{2}$ are $\{0, \frac{n-1}{2}\}$ and $\{\frac{n-1}{2}, n-1\}$ ($r(\frac{n-3}{4}) = \frac{n-3}{4}$ and $r(\frac{n-1}{2}) = \frac{n-1}{2}$).

Conversely, assume that $(n, d_1, d_2) \in \{(7, 1, 2), (7, 1, 3), (7, 2, 3), (9, 1, 3), (9, 2, 3), (9, 3, 4)\}$ and suppose that there exists a CODC of $Circ(n; \{d_1, d_2\})$ by $2P_3$.

Case 1. $(n, d_1, d_2) = (7, 1, 2)$.

Subcase 1.1. $r(1) = 1$ and $r(2) = 2$.

Assume that the edges of length 1 are $\{0, 1\}$ and $\{1, 2\}$. Edges of length 2 are $\{i, i+2\}$ and $\{i+2, i+4\}$ for some $i \in \{0, 1, 2, 3, 4, 5, 6\}$. For any i, the edge-induced subgraph induced by these four edges is not isomorphic to $2P_3$.

Subcase 1.2. $r(1) = 2$ and $r(2) = 1$.

Assume that the edges of length 1 are $\{0, 1\}$ and $\{2, 3\}$. Edges of length 2 are $\{i, i+2\}$ and $\{i+1, i+3\}$ for some $i \in \{0, 1, 2, 3, 4, 5, 6\}$. For any i, the edge-induced subgraph induced by these four edges is not isomorphic to $2P_3$.

Hence, there is no CODC of $Circ(7; \{1, 2\})$ by $2P_3$. There is no CODC of $Circ(7; \{1, 3\})$ by $2P_3$ and no CODC of $Circ(7; \{2, 3\})$ by $2P_3$, since $Circ(7; \{1, 3\}) \cong Circ(7; \{1, 2\}) \cong Circ(7; \{2, 3\})$.

Case 2. $(n, d_1, d_2) = (9, 2, 3)$.

Subcase 2.1. $r(2) = 2$ and $r(3) = 3$.

Assume that the edges of length 2 are $\{0, 2\}$ and $\{2, 4\}$. Edges of length 3 are $\{i, i+3\}$ and $\{i+3, i+6\}$ for some $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For any i, the edge-induced subgraph induced by these four edges is not isomorphic to $2P_3$.

Subcase 2.2. $r(2) = 3$ and $r(3) = 2$.

Assume that the edges of length 2 are $\{0, 2\}$ and $\{3, 5\}$. Edges of length 3 are $\{i, i+3\}$ and $\{i+2, i+5\}$ for some $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For any i, the edge-induced subgraph induced by these four edges is not isomorphic to $2P_3$.

Hence, there is no CODC of $Circ(9; \{2, 3\})$ by $2P_3$. There is no CODC of $Circ(9; \{1, 3\})$ by $2P_3$ and no CODC of $Circ(9; \{3, 4\})$ by $2P_3$, since $Circ(9; \{1, 3\}) \cong Circ(9; \{2, 3\}) \cong Circ(9; \{3, 4\})$. □

Theorem 2.5. *Suppose that $n \geq 5$. There exists a CODC of $Circ(n; \{d_1, d_2\})$ by P_5.*

Proof. *Case* 1. $2d_1 \neq d_2$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_1, 2d_1\}$; ones of length d_2 are $\{0, d_2\}$ and $\{d_2, 2d_2\}$ ($r(d_i) = d_i, i \in \{1, 2\}$).

Case 2. $2d_1 = d_2$.

Subcase 2.1. $(d_1, d_2) \notin \{(\frac{n}{5}, \frac{2n}{5}), (\frac{n}{6}, \frac{n}{3})\}$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_1, 2d_1\}$; ones of length d_2 are $\{n-4d_1, n-2d_1\}$ and $\{n-2d_1, 0\}$ ($r(d_i) = d_i, i \in \{1, 2\}$).

Subcase 2.2. $(d_1, d_2) = (\frac{n}{5}, \frac{2n}{5})$.

Edges of length $\frac{n}{5}$ are $\{\frac{3n}{5}, \frac{4n}{5}\}$ and $\{0, \frac{n}{5}\}$; ones of length $\frac{2n}{5}$ are $\{\frac{4n}{5}, \frac{n}{5}\}$ and $\{0, \frac{2n}{5}\}$ ($r(\frac{n}{5}) = \frac{2n}{5}$ and $r(\frac{2n}{5}) = \frac{n}{5}$).

Subcase 2.3. $(d_1, d_2) = (\frac{n}{6}, \frac{n}{3})$.

Edges of length $\frac{n}{6}$ are $\{\frac{2n}{3}, \frac{5n}{6}\}$ and $\{0, \frac{n}{6}\}$; ones of length $\frac{n}{3}$ are $\{\frac{5n}{6}, \frac{n}{6}\}$ and $\{0, \frac{n}{3}\}$ ($r(\frac{n}{6}) = \frac{n}{3}$ and $r(\frac{n}{3}) = \frac{n}{6}$). □

Theorem 2.6. *Suppose that $n \geq 7$. A CODC of $Circ(n; \{d_1, d_2\})$ by $P_3 + 2K_2$ exists if and only if $(d_1, d_2) \notin \{(\frac{n}{6}, \frac{n}{3}), (\frac{n}{5}, \frac{2n}{5})\}$.*

Proof. If there exists a CODC of $Circ(n; \{d_1, d_2\})$ by $P_3 + 2K_2$, then $r(d_1) = d_2$ and $r(d_2) = d_1$.

First, assume that $(d_1, d_2) \notin \{(\frac{n}{6}, \frac{n}{3}), (\frac{n}{5}, \frac{2n}{5})\}$.

Case 1. $2d_1 \neq d_2$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_2, d_2 + d_1\}$; ones of length d_2 are $\{d_2 - d_1, 2d_2 - d_1\}$ and $\{d_2, 2d_2\}$.

Case 2. $2d_1 = d_2$.

Edges of length d_1 are $\{0, d_1\}$ and $\{2d_1, 3d_1\}$; ones of length $2d_1$ are $\{n - 3d_1, n - d_1\}$ and $\{n - 2d_1, 0\}$. (Since $n \neq 2d_2$, $n \neq 4d_1$. By hypothesis, $n \notin \{5d_1, 6d_1\}$.)

Conversely, assume that $(d_1, d_2) \in \{(\frac{n}{6}, \frac{n}{3}), (\frac{n}{5}, \frac{2n}{5})\}$.

Case 1. $(d_1, d_2) = (\frac{n}{6}, \frac{n}{3})$.

Assume that the edges of length $\frac{n}{6}$ are $\{0, \frac{n}{6}\}$ and $\{\frac{n}{3}, \frac{n}{2}\}$. Edges of length $\frac{n}{3}$ are $\{i, i + \frac{n}{3}\}$ and $\{i + \frac{n}{6}, i + \frac{n}{2}\}$ for some $i \in \{0, 1, 2, \dots, n-1\}$. For $i = 0$, the edge-induced subgraph induced by these four edges is isomorphic to C_4; for $i \in \{\frac{n}{6}, \frac{5n}{6}\}$, it is isomorphic to P_5; for $i \in \{\frac{n}{3}, \frac{2n}{3}\}$, it is isomorphic to $P_4 + K_2$; for $i = \frac{n}{2}$, it is isomorphic to $2P_3$; otherwise it is isomorphic to $4K_2$. Hence, there is no CODC of $Circ(n; \{\frac{n}{6}, \frac{n}{3}\})$ by $P_3 + 2K_2$.

Case 2. $(d_1, d_2) = (\frac{n}{5}, \frac{2n}{5})$.

Assume that the edges of length $\frac{n}{5}$ are $\{0, \frac{n}{5}\}$ and $\{\frac{2n}{5}, \frac{3n}{5}\}$. Edges of length $\frac{2n}{5}$ are $\{i, i + \frac{2n}{5}\}$ and $\{i + \frac{n}{5}, i + \frac{3n}{5}\}$ for some $i \in \{0, 1, 2, \dots, n - 1\}$. For $i = 0$, the edge-induced subgraph induced by these four edges is isomorphic to C_4; for $i \in \{\frac{n}{5}, \frac{2n}{5}, \frac{3n}{5}, \frac{4n}{5}\}$, it is isomorphic to P_5; otherwise it is isomorphic to $4K_2$. Hence, there is no CODC of $Circ(n; \{\frac{n}{5}, \frac{2n}{5}\})$ by $P_3 + 2K_2$. □

Theorem 2.7. *Suppose that $n \geq 6$. A CODC of $Circ(n; \{d_1, d_2\})$ by $P_4 + K_2$ exists if and only if $(d_1, d_2) \neq (\frac{n}{5}, \frac{2n}{5})$.*

Proof. If there exists a CODC of $Circ(n; \{d_1, d_2\})$ by $P_4 + K_2$, then $r(d_1) = d_2$ and $r(d_2) = d_1$.

First, assume that $(d_1, d_2) \neq (\frac{n}{5}, \frac{2n}{5})$.

Case 1. $2d_1 \neq d_2$ and $2d_1 + d_2 \neq n$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_2, d_2 + d_1\}$; ones of length d_2 are $\{n - d_1, d_2 - d_1\}$ and $\{0, d_2\}$.

Case 2. $2d_1 = d_2$.

Edges of length d_1 are $\{n - 2d_1, n - d_1\}$ and $\{0, d_1\}$; ones of length d_2 are $\{0, 2d_1\}$ and $\{d_1, 3d_1\}$. (Since $d_2 \neq \frac{n}{2}$, $4d_1 \neq n$. By hypothesis, $5d_1 \neq n$.)

Case 3. $2d_1 + d_2 = n$.

Edges of length d_1 are $\{2d_1, 3d_1\}$ and $\{0, d_1\}$; ones of length d_2 are $\{n - 2d_1, 0\}$ and $\{n - d_1, d_1\}$. (Since $d_1 \neq d_2$, $d_2 \neq \frac{n}{2}$ and $d_2 < \frac{n}{2}$, we have $3d_1 \neq n$, $4d_1 \neq n$ and $5d_1 \neq n$, respectively.)

Conversely, assume that $(d_1, d_2) = (\frac{n}{5}, \frac{2n}{5})$. By the proof of Case 2 of the converse part of Theorem 2.6, there is no CODC of $Circ(n; \{\frac{n}{5}, \frac{2n}{5}\})$ by $P_4 + K_2$. □

Theorem 2.8. *Suppose that $n \geq 5$. A CODC of $Circ(n; \{d_1, d_2\})$ by $C_2(1, 2)$ exists if and only if $(d_1, d_2) \neq (\frac{n}{5}, \frac{2n}{5})$.*

Proof. If there exists a CODC of $Circ(n; \{d_1, d_2\})$ by $C_2(1, 2)$, then $r(d_i) = d_i$, $i \in \{1, 2\}$.

First, assume that $(d_1, d_2) \neq (\frac{n}{5}, \frac{2n}{5})$.

Case 1. $2d_1 + d_2 \neq n$ and $2d_1 \neq d_2$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_1, 2d_1\}$; ones of length d_2 are $\{n - d_2, 0\}$ and $\{0, d_2\}$.

Case 2. $2d_1 + d_2 = n$ or $2d_1 = d_2$.

Edges of length d_1 are $\{n - d_1, 0\}$ and $\{0, d_1\}$; ones of length d_2 are $\{0, d_2\}$ and $\{d_2, 2d_2\}$. (If $d_1 + 2d_2 = n = 2d_1 + d_2$, then $d_1 = d_2$, a contradiction. If $d_1 + 2d_2 = n$ and $2d_1 = d_2$, then $(d_1, d_2) = (\frac{n}{5}, \frac{2n}{5})$, a contradiction.)

Conversely, suppose that there exists a CODC of $Circ(n; \{\frac{n}{5}, \frac{2n}{5}\})$ by $C_2(1, 2)$. Assume that the edges of length $\frac{n}{5}$ are $\{0, \frac{n}{5}\}$ and $\{\frac{n}{5}, \frac{2n}{5}\}$. Edges of length $\frac{2n}{5}$ are $\{i, i + \frac{2n}{5}\}$ and $\{i + \frac{2n}{5}, i + \frac{4n}{5}\}$ for some $i \in \{0, 1, 2, \dots, n - 1\}$. For $i = \frac{4n}{5}$, the edge-induced subgraph induced by these four edges is isomorphic to $K_{1,4}$; for $i \in \{0, \frac{n}{5}, \frac{2n}{5}, \frac{3n}{5}\}$, it is isomorphic to H; otherwise it is isomorphic to $2P_3$. Hence, there is no CODC of $Circ(n; \{\frac{n}{5}, \frac{2n}{5}\})$ by $C_2(1, 2)$. □

Theorem 2.9. *Suppose that $n \geq 5$. There exists a CODC of $Circ(n; \{d_1, d_2\})$ by C_4.*

Proof. Edges of length d_1 are $\{0, d_1\}$ and $\{d_2, d_1 + d_2\}$; ones of length d_2 are $\{0, d_2\}$ and $\{d_1, d_2 + d_1\}$ ($r(d_1) = d_2$ and $r(d_2) = d_1$). □

Theorem 2.10. *Suppose that $n \geq 5$. A CODC of $Circ(n; \{d_1, d_2\})$ by H exists if and only if $n \in \{2d_1 + d_2, d_1 + 2d_2\}$ or $2d_1 = d_2$.*

Proof. If there exists a CODC of $Circ(n; \{d_1, d_2\})$ by H, then $r(d_i) = d_i$, $i \in \{1, 2\}$.

First, assume that there exists a CODC of $Circ(n; \{d_1, d_2\})$ by H. We consider two possibilities. If two edges of the triangle in H are of length d_1, then either $2d_1 = d_2$ or $n = 2d_1 + d_2$. If two edges of the triangle in H are of length d_2, then $n = d_1 + 2d_2$. Hence, $n \in \{2d_1 + d_2, d_1 + 2d_2\}$ or $2d_1 = d_2$.

Conversely, assume that $n \in \{2d_1 + d_2, d_1 + 2d_2\}$ or $2d_1 = d_2$.

Case 1. $d_1 + 2d_2 = n$.

Edges of length d_1 are $\{2d_2, 0\}$ and $\{0, d_1\}$; ones of length d_2 are $\{0, d_2\}$ and $\{d_2, 2d_2\}$.

Case 2. $2d_1 = d_2$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_1, 2d_1\}$; ones of length d_2 are $\{n - d_2, 0\}$ and $\{0, d_2\}$.

Case 3. $2d_1 + d_2 = n$.

Edges of length d_1 are $\{0, d_1\}$ and $\{d_1, 2d_1\}$; ones of length d_2 are $\{2d_1, 0\}$ and $\{0, d_2\}$. $\square$

Acknowledgements

The first author was supported by the Department of Science and Technology (DST), Government of India, New Delhi; Project Grant No. SR/S4/MS: 481/07. The second author was supported by a UGC Research Fellowship; Grant No. F4-1/2008 (BSR). The authors thank DST and UGC for the financial support.

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